

# Physics from HBT radii

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## Abstract

An approximate formula connecting the true and the HBT homogeneity regions in multiparticle production processes is derived. It implies that when calculating the HBT radii one should use the center of mass systems of the pairs rather than the now popular LCMS system. A discussion of several simple examples clarifies the potential and limitations of the HBT method. The even cumulants of the  $\mathbf{X}$ -distribution, including the HBT radii, can be determined for each homogeneity region, but the relative positions of the homogeneity regions are unconstrained. This makes the HBT radii of little use for calculating quantities dependent on the interparticle interactions in coordinate space.

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Almost fifty years ago, in a famous paper known as GGLP [1], a method of using momentum distributions for pairs of identical pions to estimate the

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sizes of the interaction regions, i.e. of the regions where the hadrons are produced in multiparticle production processes, has been described. Such radii, determined from momentum distributions of identical particles, have been later called, not very appropriately [2], HBT radii. GGLP assumed that hadron production happens instantly and simultaneously at some time  $t = 0$  and that there are no correlations between the momenta of the hadrons and their production points. Considering simultaneously the momentum of a particle and its production point implies a quasiclassical approximation, but it can be made plausible [3] that this approximation is good for heavy ion scattering and acceptable also for the other multiparticle production processes. Let us denote by  $p_1$  and  $p_2$  the four-momenta of the two identical pions in the pair<sup>1</sup>, by  $x_1$  and  $x_2$  the space-time positions of their production points and introduce the notation

$$K = \frac{1}{2}(p_1 + p_2); \quad q = p_1 - p_2; \quad X = \frac{1}{2}(x_1 + x_2); \quad y = x_1 - x_2. \quad (1)$$

Note that  $p_1^2 = p_2^2 = m^2$  implies

$$K_0 = \frac{1}{\sqrt{2}} \sqrt{E_k^2 + \frac{1}{4}\mathbf{q}^2 + \sqrt{(E_k^2 + \frac{1}{4}\mathbf{q}^2)^2 - (\mathbf{K} \cdot \mathbf{q})^2}}; \quad q_0 = \boldsymbol{\beta} \cdot \mathbf{q}, \quad (2)$$

where  $E_k = \sqrt{\mathbf{K}^2 + m^2}$  equals  $K_0$  at  $\mathbf{q} = 0$ , and  $\boldsymbol{\beta} = \frac{\mathbf{K}}{K_0}$  is the velocity of the pair. GGLP found  $R_{HBT}$  from the distribution of  $q^2$ .

Some twenty five year later Pratt [4] described a model with an exploding source. In this model, the momentum distribution of particles depends on the production point. Due to this correlation, the HBT radius differs significantly from the true radius, which is known from the input. For fixed  $\mathbf{K}$  the result was that, while the true radius  $R$  does not depend on  $|\mathbf{K}|$ , the HBT radius decreases from  $\sqrt{\frac{2}{3}}R$  for  $\mathbf{K} = 0$  to zero when  $|\mathbf{K}|$  tends to infinity. This finding was generalized by Bowler [5], who pointed out that in general, whenever there are strong correlations between momenta and production points,  $R_{HBT}$  measures only the region where particles of similar momentum are produced. Sinyukov [6], [7] associated these regions with the homogeneity

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<sup>1</sup>We discuss pions for definiteness, but the discussion applies to any spin zero bosons.

regions considered in hydrodynamics and the name homogeneity region got generally accepted.

Most models in their simplest form, i.e. without final state interactions, resonance decays etc., find the  $n$ -particle momentum distributions for identical pions from formulae equivalent to [8]

$$P(\mathbf{p}_1, \dots, \mathbf{p}_n) = C_n \sum_Q \prod_{j=1}^n \rho(\mathbf{p}_j; \mathbf{p}_{Qj}), \quad (3)$$

where the summation is over all the permutations  $j \rightarrow Qj$  of the indices  $j = 1, \dots, n$ ,  $C_n$  are normalization constants and  $\rho(\mathbf{p}_1; \mathbf{p}_2)$  is some time independent, single particle density matrix. Thus  $\rho(\mathbf{p}_1; \mathbf{p}_2)$  determines all the momentum distributions. The best [9] quantum mechanical analogue of the classical phase space density is the Wigner function  $W(\mathbf{X}, \mathbf{K})$ , which is a Fourier transform of  $\rho(\mathbf{p}_1; \mathbf{p}_2)$ . Following most models, we assume further that,  $W(\mathbf{X}, \mathbf{K})$  defines the HBT  $\mathbf{X}$ -distribution for the homogeneity region corresponding to a given  $\mathbf{K}$ .

Measuring all the momentum distributions is not enough to determine  $\rho(\mathbf{p}_1; \mathbf{p}_2)$ . As easily seen from (3), the observable momentum distributions do not change [10], [11], under the transformation<sup>2</sup>

$$\rho(\mathbf{p}_1; \mathbf{p}_2) \rightarrow \rho_f(\mathbf{p}_1; \mathbf{p}_2) = e^{i[f(p_1) - f(p_2)]} \rho(\mathbf{p}_1; \mathbf{p}_2), \quad (4)$$

where  $f(p)$  is an arbitrary real-valued function of the four-vector  $p$ , and  $p_0 = \sqrt{\mathbf{p}^2 + m^2}$ . This transformation does not affect the HBT radii<sup>3</sup> of the homogeneity regions, but it can shift and deform these regions.

An important concept introduced in [4] is the emission function  $S(X, K)$ . Assuming chaotic sources, i.e. no interference between particles produced at different moments of time, this is related to the density matrix in the momentum representation by the formula<sup>4</sup>

<sup>2</sup>The full group of transformations includes also  $\rho(\mathbf{p}_1; \mathbf{p}_2) \rightarrow \rho^*(\mathbf{p}_1; \mathbf{p}_2)$ , but this corresponds to the space inversion of the interaction region and is of no interest in the present context.

<sup>3</sup>More generally, all the even cumulants of  $W(\mathbf{X}, \mathbf{K})$ , at  $\mathbf{K}$  fixed, remain unchanged. The proof is a simple extension of the proof given for the HBT radii in [11].

<sup>4</sup>Sometimes formulae equivalent to (5) with  $\rho(\mathbf{p}_1; \mathbf{p}_2)$  replaced by  $\rho^*(\mathbf{p}_1; \mathbf{p}_2)$  are used. The advantage of the present convention is that for  $S(X, K) = \delta(t)g(\mathbf{X}, \mathbf{K})$  function  $g$  is just the Wigner function.

$$\rho(\mathbf{p}_1; \mathbf{p}_2) = \int d^4 X S(X, K) e^{iqX}. \quad (5)$$

In the spirit of the quasiclassical approximation, using the well tested (cf. e.g. [12] and references given there) mass shell approximation

$$K_0 = E_k \equiv \sqrt{m^2 + \mathbf{K}^2}, \quad (6)$$

one can interpret  $S(X, K)$  as the time-dependent distribution of the pairs of vectors  $\{\mathbf{X}, \mathbf{K}\}$ . The time independent homogeneity region for each  $\mathbf{K}$  can be obtained by integrating over time

$$p(\mathbf{X}|\mathbf{K}) = \int dt S(X, K), \quad (7)$$

where the notation stresses that we are interested in the distribution of  $\mathbf{X}$  for given  $\mathbf{K}$ . At this point the mass shell approximation (6) is necessary in order to make  $K_0$ , and consequently  $S(X; K)$  and  $p(\mathbf{X}|\mathbf{K})$ , independent of  $\mathbf{q}$ .

Further we will call  $p(\mathbf{X}|\mathbf{K})$  true distribution in order to distinguish it from the HBT distribution given by the Wigner function. Invariance (4) means that for a given set of momentum distributions there is a variety of HBT  $\mathbf{X}$ -distributions which all correspond to the same fit to the data. One way of choosing among them is to specify the emission function.

It was soon noticed [13] that for a given  $\rho(\mathbf{p}_1; \mathbf{p}_2)$  there is an infinity of solutions for  $S$  and that, in particular, a small long-lived source may be undistinguishable from a large short-lived source. Thus, the information about interaction regions obtained from momentum measurements is rather incomplete.

The standard relation between the density matrix and the Wigner function yields from (5)

$$W(\mathbf{X}, \mathbf{K}) = \int \frac{d^3 q}{(2\pi)^3} \int d^4 X' S(X', K) e^{i\mathbf{q} \cdot (\mathbf{X} - \mathbf{X}' + \boldsymbol{\beta}t)}, \quad (8)$$

where  $q_0$  has been eliminated using (2). We invoke now the mass shell approximation (6), which makes  $S$  independent of  $\mathbf{q}$ . Thus, the integration  $d^3 q$  gives  $(2\pi)^3 \delta(\mathbf{X} - \mathbf{X}' + \boldsymbol{\beta}t)$  and the integration  $d^3 X'$  can be performed. The result is

$$W(\mathbf{X}, \mathbf{K}) = \int dt S(\mathbf{X} + \boldsymbol{\beta}t, t, K). \quad (9)$$

The comparison of the true interaction region with the HBT one reduces to the comparison of the integrals in (7) and (9).

The integrals in (7) and (9), and consequently the true and HBT homogeneity regions, coincide only<sup>5</sup> for  $\boldsymbol{\beta} = \mathbf{0}$ . This can be achieved by working with pairs which have the same velocity  $\boldsymbol{\beta}$  and using the reference frame where  $\boldsymbol{\beta} = \mathbf{0}$ .

We summarize our findings:

- The HBT results are credible only for single homogeneity regions, i.e. at given  $\mathbf{K}$ .
- For each homogeneity region, only the pairs with the same velocity, i.e. with the same  $\mathbf{K}$ , are considered and one should use the reference frame, where this velocity is zero. There, the true and the HBT homogeneity regions coincide which is not the case for other frames, as for instance for the now popular (cf e.g. the review [14]) LCMS frame. An additional pragmatic argument in favor of the rest frame is that the effects of the final state interactions are the simplest there (cf. e.g. [17] and the references given there).
- The true radii of the homogeneity regions, more generally all the even cumulants of their  $\mathbf{X}$ -distributions, can be obtained from the HBT analysis of the momentum distributions. For instance, when  $p(\mathbf{X}|\mathbf{K}) = p(-\mathbf{X}|\mathbf{K})$  all the odd cumulants vanish and, therefore,  $p(\mathbf{X}|\mathbf{K})$  can be measured by the HBT method.
- The space distribution of the centers of the homogeneity regions  $\langle \mathbf{X} \rangle(\mathbf{K})$  is unconstrained by the HBT analysis, but once it is fixed, all the functions  $p(\mathbf{X}|\mathbf{K})$  can be determined [10], [11].

Let us discuss some simple examples illustrating these features of the HBT method. In order to avoid unessential complications, we will consider one space dimension and very simple emission functions constructed from  $\delta$ -functions and step functions. They violate the Heisenberg uncertainty relations, but all these calculations can be repeated in three dimensions using Gaussians and the results are qualitatively the same.

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<sup>5</sup>Except for the trivial case when  $S(X, K) \sim \delta(t)$ .

Let us first consider the emission function

$$S_1(X, K; t_0, a, b, \kappa) = \delta(t - t_0) \Theta_{a,b}(\mathbf{X}) \delta(\mathbf{K} - \kappa), \quad (10)$$

where  $t_0$ ,  $a < b$  and  $\kappa$  are real constants. Function  $\Theta_{a,b}(x) = \frac{1}{b-a}$  for  $a < x < b$  and zero outside this interval. For this emission function

$$\rho(\mathbf{p}_1; \mathbf{p}_2) = \delta(\mathbf{K} - \kappa) e^{-i\mathbf{q}(\frac{a+b}{2} - \beta t_0)} \frac{2 \sin |\mathbf{q}| \frac{b-a}{2}}{|\mathbf{q}|(b-a)} \quad (11)$$

$$W(\mathbf{X}, \mathbf{K}) = \Theta_{a,b}(\mathbf{X} + \beta t_0) \delta(\mathbf{K} - \kappa). \quad (12)$$

It is seen that the segment  $a < \mathbf{X} < b$  got shifted to  $a - \beta t_0 < \mathbf{X} < b - \beta t_0$ . According to (11) this shift is due to the phase factor in the density matrix. Since in the formula for the two-particle correlation function only the absolute value of  $\rho$  appears, experiment is blind to such shifts. This remains true also when many-particle correlation functions are measured, even when all the measurements are performed with perfect precision [10], [11]. The fact that the position of the center of the interaction region cannot be found from the HBT analysis of the momentum distributions is, of course, well known (see e.g. [12]), but one should also keep in mind that when the emission function is given the density matrix, the Wigner function and, consequently, all the HBT homogeneity regions are unambiguously defined.

As the next example we take

$$S_2(X, K) = \frac{1}{2} [S_1(X, K; t_0, -a, 0, \kappa_1) + S_1(X, K; t_0, 0, a, \kappa_2)] \quad (13)$$

Using the previous example, it is seen that the segment  $-a < \mathbf{X} < a$  is mapped onto two segments:  $-a - \beta_1 t_0 < \mathbf{X} < -\beta_1 t_0$  and  $-\beta_2 t_0 < \mathbf{X} < a - \beta_2 t_0$ . Let us choose

$$f(p) = -\mathbf{b} \cdot \mathbf{p} - \frac{1}{2} c \mathbf{p}^2, \quad (14)$$

where  $\mathbf{b}$  is an arbitrary vector and  $c$  an arbitrary constant, and make the transformation (4). Each HBT homogeneity region gets shifted by  $\mathbf{b} + c\mathbf{K}$ . For  $\kappa_1 \neq \kappa_2$ , by a suitable choice of  $c$ , one can obtain any prescribed distance between the centers of the two segments. The true length of the interaction region, as seen from  $S$ , is  $2a$ . The length of the transformed interaction region, which follows just as well from the data on momentum distributions,

can be any number not smaller than  $a$ . What is the way out? One has to invoke the homogeneity regions. For each  $\mathbf{K}$  separately, the length of the segment where particles with this value of  $\mathbf{K}$  are produced, i.e. of the homogeneity region, is reproduced correctly. The positioning of the homogeneity regions corresponding to different values of  $\mathbf{K}$ , however, is beyond control when the momentum distributions are the only input. In three dimensions it is easy to prove [10], [11] that by a suitable choice of function  $f(p)$  in (4), the positions of the centers of the homogeneity regions  $\langle \mathbf{X} \rangle(\mathbf{K})$  can be changed into  $\langle \mathbf{X} \rangle(\mathbf{K}) + \mathbf{g}(\mathbf{K})$ , where  $\mathbf{g}$  is an arbitrary differentiable function of  $\mathbf{K}$ .

As an amusing example in three dimensions let us consider the models where  $K_\mu \approx \lambda X_\mu$ , and  $\lambda$  is a constant [15], [16]. They correspond to emission functions

$$S(X, K) = \delta^3(\mathbf{X} - \beta t) \bar{S}(\mathbf{X}, t, K), \quad (15)$$

where  $\bar{S}$  is some function which does not affect the singularity introduced by the  $\delta^3$ . The corresponding Wigner function is

$$W(\mathbf{X}, \mathbf{K}) = \delta^3(\mathbf{X}) \int dt \bar{S}(\beta t, t, K). \quad (16)$$

Thus, the HBT interaction region reduces to one point. To be sure: such models, when properly used, are quite successful, but finding the interaction region from the density matrix in the momentum representation is their misuse.

As our last model consider the emission function

$$S(X, K) = \Theta_{0,a}(t) \delta(\mathbf{X}) \delta(\mathbf{K} - \kappa). \quad (17)$$

This corresponds to the Wigner function

$$W(\mathbf{X}, \mathbf{K}) = \frac{1}{\beta} \Theta_{0,a}(-\mathbf{X}/\beta) \delta(\mathbf{K} - \kappa) = \Theta_{-\beta a, 0}(\mathbf{X}) \delta(\mathbf{K} - \kappa). \quad (18)$$

The length of the true interaction region is zero, while the length of the HBT interaction region is  $\beta a$ . Thus, in order to get the true length from the HBT analysis one must use the reference frame where  $\mathbf{K} = \mathbf{0}$ . We conclude that, working with all the pairs which have a given velocity, one should measure their homogeneity region in their rest frame. Let us consider some further implications of our analysis:

If there are no position-momentum correlations, all the homogeneity regions, measured in the respective rest frames, are equivalent to each other and to the overall interaction region. The ambiguity (4) reduces to a lack of information about the position of the center of the interaction region [11]. Thus, in this case the HBT method works very well. One should keep in mind however that, as illustrated by our second model, the independence of the homogeneity region on  $\mathbf{K}$  is a necessary, but not a sufficient condition for the absence of position-momentum correlations.

If there are no interparticle interactions in coordinate space, the HBT homogeneity regions may be all we need. E.g. the entropy of a gas of noninteracting particles (in the quasiclassical approximation) depends on the accessible phase space volume. This can be calculated by integrating over coordinate space at fixed momentum and then integrating over momenta. The first integration gives just the HBT volume of the homogeneity region, though one should keep in mind the ambiguity (4). If there are interparticle interactions in coordinate space, however, it may make a lot of difference whether the homogeneity regions are on top of each other, or scattered over space. Thus, the use of the HBT radii to calculate the entropy of a gas of interacting particles, or of their mean free paths, is risky.

For  $\beta = 0$  the true space density  $\int dt S(X, K)$  can be unambiguously obtained from  $\rho(\mathbf{p}_1, \mathbf{p}_2)$  by inverse Fourier transformation. Thus, all the ambiguity in the determination of the homogeneity regions, including their absolute positions, results from (4) and disappears when  $f(p)$  is fixed. Conversely, when the true space density is known, it yields unambiguously  $\rho(\mathbf{p}_1; \mathbf{p}_2)$  and thus it fixes  $f(p)$ .

The ambiguity in the determination of the space-time density  $S$  follows from the irreversibility of transformation (5) and persists even when  $f(p)$  is fixed, i.e. when  $\rho(\mathbf{p}_1; \mathbf{p}_2)$  is known.

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